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Transition systems without transitions ☆

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Abstract

We study the problem of embedding partial 2-structures into set 2-structures such that the target structure is full and forward closed and it is minimal w.r.t. these properties.

Ehrenfeucht and Rozenberg introduced the notion of partial 2-structures—an abstract form of transition systems—and studied the problem of representing them as partial set 2-structures—directed graphs made of sets and their ordered symmetric differences. They constructed a representation and gave the conditions under which the representation is an isomorphism.

We propose an alternative representation of partial 2-structures by partial set 2-structures which are complete graphs, hence their transitions may be left implicit, yielding a static representation of dynamic systems.

Keywords: 2-Structure; Transition system; Elementary net; Concurrency theory

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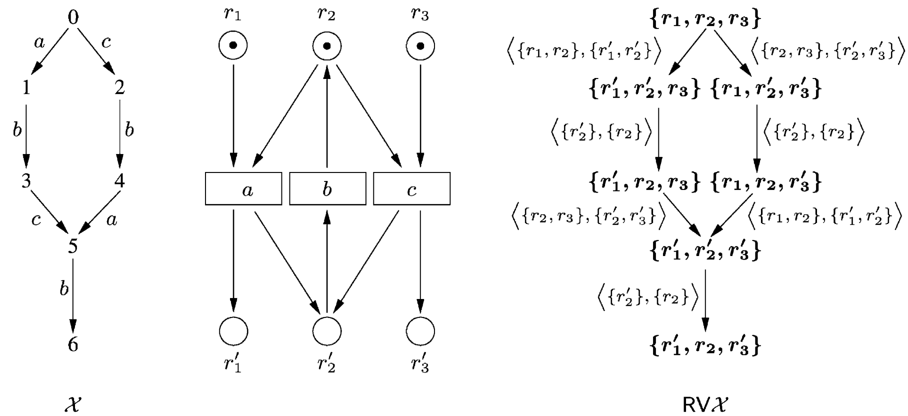

Fig. 1.

1. Introduction

Andrzej Ehrenfeucht and Grzegorz Rozenberg introduced in the early 1990s the notion of (*partial*) 2-*structures*—an abstract form of directed graphs with labelled edges—and studied the problem of representing them as partial *set* 2-structures—directed graphs whose nodes are sets and whose edges are labelled with ordered symmetric differences of sets. This problem has a strong relationship with the problem of representing graphs as case graphs of elementary net systems. The above authors constructed a partial set 2-structure representation of partial 2-structures and they exhibited the conditions under which the representation is an isomorphism. These conditions are, for instance satisfied in the partial 2-structure $\mathcal{X}$ shown on the left-hand side of Fig. 1, where labels indicate classes of equivalent edges. This partial 2-structure has six non-trivial regions r_1, r_2, r_3 and r'_1, r'_2, r'_3, where $r_1 = \{0, 2, 4\}$, $r_2 = \{0, 3, 4, 6\}$, $r_3 = \{0, 1, 3\}$ and the remaining regions are the respective complements of the latter (plus two trivial regions $\{0, \ldots, 6\}$ and $\emptyset$). The partial set 2-structure $\mathsf{RV}\mathcal{X}$ (the regional version of $\mathcal{X}$) is shown on the right-hand side of Fig. 1 (only the non-trivial regions are shown). It may be seen as a Kripke structure where the propositions are the regions of $\mathcal{X}$ and where different nodes satisfy different sets of propositions. Edges represent occurrences of logical changes, each of which is defined by the ordered symmetric difference between two sets of propositions. Two edges are equivalent if they represent two occurrences of the same change, i.e. if the ordered symmetric difference between their source and target nodes is the same. For instance, all images of edges in the equivalence class b are occurrences of the same change $\langle\{r'_2\}, \{r_2\}\rangle$. The partial set 2-structure $\mathsf{RV}\mathcal{X}$ is moreover isomorphic to the reachable case graph of the elementary net system shown at the centre of Fig. 1.

In the present paper, we aim at a similar goal, but by using *full* substructures of set 2-structures as representations. Thus, our goal is to represent a partial 2-structure with a set of *logical states* defined as sets of propositions, such that the axioms given by Ehrenfeucht and Rozenberg hold in the *complete* graph made of these logical states (as nodes) and their ordered pairs (as edges). The label of an edge is the ordered symmetric difference between

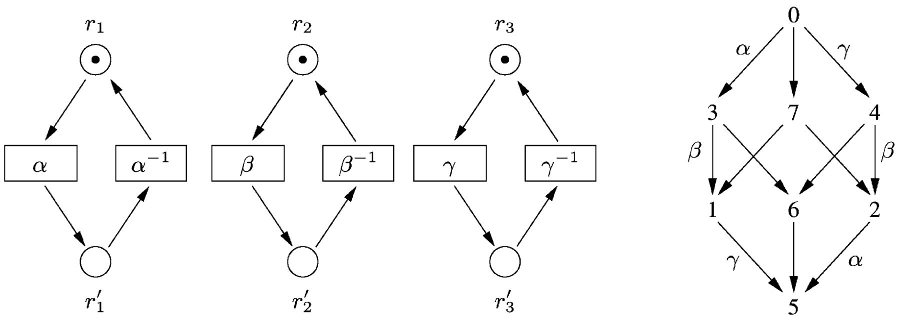

Fig. 2.

its end points. In this static representation of dynamic systems, the edges or transitions are left totally implicit: every ordered pair of logical states is an implicit edge. In particular, all edges have reverse edges and the edges form a partial group under concatenation. As two edges labelled with the same ordered difference are two occurrences of the same change, a partial group of logical changes derives through a quotient. Some logical changes represent classes of edges of the original partial 2-structure. They form a set of generators for all other changes in the partial group, provided the original partial 2-structure is connected.

In order to illustrate these ideas, let us consider the partial set 2-structure $\mathsf{RV}\mathcal{X}$ on the right-hand side of Fig. 1. $\mathsf{RV}\mathcal{X}$ is a substructure of the set 2-structure, whose base set is formed by r_1, r_2, r_3 and their complements. However, this substructure is not *full*. For instance, there is no edge from the logical state $\{r_1, r_2, r'_3\}$ to the logical state $\{r_1, r'_2, r'_3\}$. The missing edge is an instance of the logical change $\langle\{r_2\}, \{r'_2\}\rangle$ which represents b^{-1}, since the reverse logical change $\langle\{r'_2\}, \{r_2\}\rangle$ represents b. Now suppose this missing edge has been added. According to the *forward-closure* axiom given by Ehrenfeucht and Rozenberg, the logical change $\langle\{r_2\}, \{r'_2\}\rangle$ may then also occur in the logical state $\{r_1, r_2, r_3\}$. When it occurs in $\{r_1, r_2, r_3\}$, the considered change leads to the logical state $\{r_1, r'_2, r_3\}$. The latter state is therefore missing in $\mathsf{RV}\mathcal{X}$. Once $\{r_1, r'_2, r_3\}$ has been added to $\mathsf{RV}\mathcal{X}$, the new set of nodes is in bijection with the product space $\{r_1, r'_1\} \times \{r_2, r'_2\} \times \{r_3, r'_3\}$ and it is closed under arbitrary changes defined by ordered symmetric differences between nodes. This product space—together with the embedding of the set of states $\{0, \ldots, 6\}$ in this product—is the static representation $\overline{\mathcal{X}}$ we propose for the partial 2-structure $\mathcal{X}$. Compared to $\mathcal{X}$, the representation $\overline{\mathcal{X}}$ has better structural properties with respect to concurrency. It coincides for instance with the set of reachable cases of the elementary net system shown together with its case graph in Fig. 2. The cases are numbered alike the nodes of $\mathcal{X}$ which they represent, respectively, with the exception of the case $7 = \{r_1, r'_2, r_3\}$ which was missing. Remarkably, α, β and γ are *independent* changes, whereas a, b and c are not. The latter may be reconstructed as $a = \alpha{\cdot}\beta$, $b = \beta^{-1}$ and $c = \gamma{\cdot}\beta$ and since $\overline{\mathcal{X}}$ is a complete graph, these expressions define actually classes of edges of $\overline{\mathcal{X}}$. Intuitively, $\mathcal{X}$ coincides with an *observable* restriction of $\overline{\mathcal{X}}$, where the restriction bears both on states (7 is unobservable) and on classes of transitions (only $a = \alpha{\cdot}\beta$, $b = \beta^{-1}$, and $c = \gamma{\cdot}\beta$ are observable in the partial group generated by α, β and γ, or yet by a, b, c).

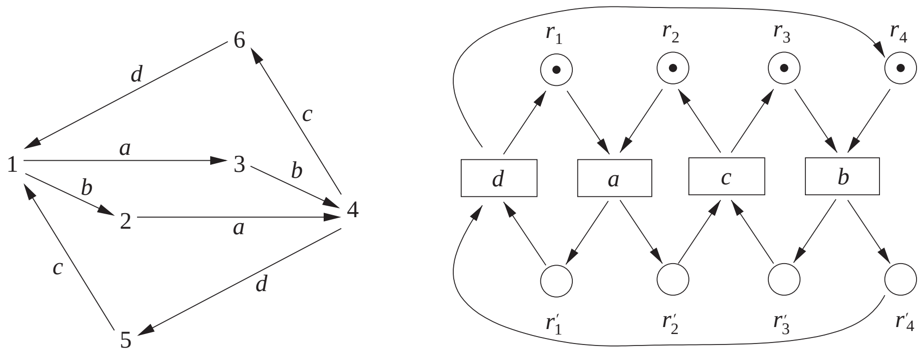

Fig. 3.

The work presented herein has a loose connexion with the work by Luca Bernardinello, Carlo Ferigato and Lucia Pomello. These authors proposed another representation of partial set 2-structures as full substructures of set 2-structures. Both representations coincide for the partial 2-structure $\mathcal{X}$ from Fig. 1, but they differ for the partial 2-structure shown on the left-hand side of Fig. 3 (borrowed from [1, Fig. 8]). The partial set 2-structure representation given in [1] is a product space $\{r_1, r'_1\} \times \cdots \times \{r_4, r'_4\}$, where $r_1 = \{1, 2, 5\}$, $r_2 = \{1, 2, 6\}$, $r_3 = \{1, 3, 6\}$, $r_4 = \{1, 3, 5\}$ and the other regions are their respective complements. The partial set 2-structure representation which we suggest has exactly 6 nodes—the images of the original nodes suffice. To give a hint at the difference, the construction defined in [1] is an algebraic completion (of partial 2-structures), whereas we define here an *inductive* completion algorithm (of partial 2-structures). The partial 2-structure shown on the left-hand side of Fig. 3 is isomorphic to the reachable case graph of the elementary net system shown on the right-hand side of Fig. 3. In the algebraic completion, every case of the net in which one of r_i and r'_i is marked for each $i \in \{1, 2, 3, 4\}$ is taken into account. In the inductive completion, one discards the cases that cannot be reached even though the set of transitions (of the net) has been extended to a partial group.

The structure of the paper is as follows. In Section 1 we recall definitions, mainly from [2]. In Section 2 we recall the main result of [2]. In Section 3 we construct on top of this the static representation of a partial 2-structure (Theorems 3 and 4 are parallel to similar theorems in [2]). In Section 4 we examine further the properties of the construction. In Section 5 we show with examples the two different reasons why the construction is non-trivial, i.e. it does add new states. In Section 6 we sketch a comparison with the construction studied by Bernardinello et al.

2. Preliminaries

Let X, Y be sets and $f : X \to Y$ a function. We write $E_2(X)$ to denote the set $\{\langle x, y\rangle \mid x, y \in X, x \neq y\}$ and $E_2(f) : E_2(X) \rightharpoonup E_2(Y)$ to denote the partial function defined as $E_2(f)\,\langle x, y\rangle = \langle fx, fy\rangle$, provided $fx \neq fy$. If $A \subseteq X$ is a subset, then its image under f will be denoted by $f^{\rightarrow}A = \{fx \mid x \in A\}$. If $B \subseteq Y$, then its pre-image will be denoted by $f^{\leftarrow}B = \{x \in X \mid fx \in B\}$. *All sets considered in this paper are finite.*

Definition 1. A *partial* 2-*structure* is a triple $\langle X, E, \sim\rangle$, where X is a finite set, $E \subseteq E_2(X)$, $E \neq \emptyset$, and $\sim$ is an equivalence relation on E.

We say that a partial 2-structure $\langle Y, F, \approx\rangle$ is a *substructure* of $\langle X, E, \sim\rangle$ if $Y \subseteq X$, $F \subseteq E$ and $\approx\ =\ \sim \cap (F \times F)$. It is a *full substructure* if, additionally, $F = E \cap E_2(Y)$.

A 2-*structure* is a partial 2-structure $\langle X, E_2(X), \sim\rangle$.

A partial 2-structure $\langle X, E, \sim\rangle$ may be viewed as a transition system with the set of states X, the set of labels equal to $E \times E/\sim$ and transitions $\langle x, [\langle x, y\rangle]_\sim, y\rangle$, provided $\langle x, y\rangle \in E$. Conversely, if $\langle S, L, T\rangle$, $T \subset S \times L \times S$, is a finite transition system without self-loops and parallel transitions, then one may consider it as a partial 2-structure with $\langle s, s'\rangle \sim \langle s'', s'''\rangle$ *iff* $\langle s, \lambda, s'\rangle, \langle s'', \lambda, s'''\rangle \in T$ for some $\lambda \in L$. These relationships have already been illustrated in the introduction. In the sequel, the carrier set X of a partial 2-structure is called its set of states. The underlying graph of a partial 2-structure $\langle X, E, \sim\rangle$ is $\langle X, E\rangle$.

Definition 2. A *morphism* of partial 2-structures $\phi : \langle X, E, \sim\rangle \to \langle Y, F, \approx\rangle$ is a function $\phi : X \to Y$ such that if $\langle x, y\rangle \sim \langle z, w\rangle$, then either $\phi x = \phi y$ and $\phi z = \phi w$ or $\langle \phi x, \phi y\rangle \approx \langle \phi z, \phi w\rangle$.

A morphism ϕ is *injective* if $x \neq y$ implies $\phi x \neq \phi y$; it is *strongly injective* if, additionally, $\langle x, y\rangle \not\sim \langle z, w\rangle$ implies $\langle \phi x, \phi y\rangle \not\approx \langle \phi z, \phi w\rangle$.

A morphism is *surjective* if for every $y \in Y$ there is $x \in X$ such that $\phi x = y$ and *strongly surjective* if, additionally, for every $\langle z, w\rangle \in F$ there are $x, y \in X$ such that $\langle \phi x, \phi y\rangle \approx \langle z, w\rangle$.

A morphism ϕ is an *isomorphism* if it is a bijection and its inverse ϕ^{-1} is a morphism, too.

Definition 3. Let $\phi : \langle X, E, \sim\rangle \to \langle Y, F, \approx\rangle$ be a morphism. Its *image* is the substructure of $\langle Y, F, \approx\rangle$ defined as $\langle \phi^{\rightarrow}X, E_2(\phi)^{\rightarrow}E, \approx \cap\ (E_2(\phi)^{\rightarrow}E \times E_2(\phi)^{\rightarrow}E)\rangle$.

Given a morphism $\phi : \mathcal{X} \to \mathcal{Y}$ and a substructure $\mathcal{Y}_0$ of $\mathcal{Y}$ such that the image of $\mathcal{X}$ under ϕ is a substructure of $\mathcal{Y}_0$, we can consider the co-restriction of ϕ on $\mathcal{Y}_0$ that we shall denote also $\phi : \mathcal{X} \to \mathcal{Y}_0$. On the other hand, if $\mathcal{X}_0$ is a substructure of $\mathcal{X}$, we shall denote by $\phi\big|_{\mathcal{X}_0} : \mathcal{X}_0 \to \mathcal{Y}$ the restriction of ϕ to $\mathcal{X}_0$.

Definition 4. We denote by **2** the 2-structure with the set of states $\{0, 1\}$ and with the equivalence classes $\{\langle 0, 1\rangle\}$ and $\{\langle 1, 0\rangle\}$.

Definition 5. A *region* r of a partial 2-structure $\mathcal{X}$ is a morphism $r : \mathcal{X} \to \mathbf{2}$. The set of all regions of $\mathcal{X}$ is denoted by $\mathsf{Reg}_{\mathcal{X}}$.

Note that a region of a partial 2-structure is a characteristic function. We will consider it as a subset of the set of states of the partial 2-structure. Moreover, equivalent edges cross the region in a similar way: they either enter the region, or leave it, or do not cross its border. For instance, the set of states $\{0, 1, 2\}$ is not a region of the partial 2-structure $\mathcal{X}$ from Fig. 1, because the edge from 5 to 6 does not cross its border, while equivalent edges $\langle 1, 3\rangle$ and $\langle 2, 4\rangle$ do cross the border. The above definitions 4 and 5 restate equivalently the original definitions given in [2].

Definition 6. The *ordered symmetric difference* of two sets X and Y is a pair $\langle X\setminus Y, Y\setminus X\rangle$, notation $X \vartriangle Y$. A *set* 2-*structure* with base set B is a 2-structure $\langle 2^B, E_2(2^B), \triangleq\rangle$ where 2^B stands for the powerset of B and $\triangleq$ is the kernel of the function $\vartriangle$, i.e. $\langle x, y\rangle \triangleq \langle z, w\rangle$ if $x \vartriangle y = z \vartriangle w$. The set 2-structure with base set B is denoted B^*. A *partial set 2-structure* is a substructure of a set 2-structure.

Note that **2** is isomorphic to the set 2-structure with the singleton base set. A full substructure of a set 2-structure is in general *not* a set 2-structure, despite the fact that it is a 2-structure. A partial set 2-structure is deterministic in the sense that $\langle x, y\rangle \triangleq \langle x, w\rangle$ implies $y = w$. Also $\langle y, x\rangle \triangleq \langle w, x\rangle$ implies $y = w$. Consecutive arrows $\langle x, y\rangle$ and $\langle y, z\rangle$ cannot be equivalent, too.

Definition 7. Given $\mathcal{A} \subseteq 2^A$ and $\mathcal{B} \subseteq 2^B$, a function $\sigma : \mathcal{A} \to \mathcal{B}$ is said to *preserve set difference* if it may be extended (resp. co-extended) to the subset of 2^A (resp. 2^B) generated from $\mathcal{A}$ (resp. $\mathcal{B}$) using the operations of set difference and disjoint union of sets, *consistently* with the law $\sigma(x\setminus y) = \sigma x\setminus\sigma y$, $x, y \in 2^A$.

Proposition 1. *Let $\mathcal{Y} = \langle Y, E, \sim\rangle$ and $\mathcal{Z} = \langle Z, F, \approx\rangle$ be partial set 2-structures. Every function $\sigma : Y \to Z$ that preserves set difference and that sends every edge $\left\langle y, y'\right\rangle \in E$ to an edge $\langle \sigma y, \sigma y'\rangle \in F$ unless $\sigma y = \sigma y'$ defines a morphism $\sigma : \mathcal{Y} \to \mathcal{Z}$.*

Proof. Obvious from Definitions 2, 6, and 7. □

Note that a set difference preserving function is monotone w.r.t. set inclusion and maps disjoint sets to disjoint sets. In fact, a morphism σ of set 2-structures preserves set difference *iff* $\sigma\emptyset = \emptyset$ *iff* $\sigma X = \bigcup_{x\in X} \sigma\{x\}$.

Proposition 2. *Let $f : B \to A$ be a function. A pre-image map defined as $f^*x = f^{\leftarrow}x$, $x \in 2^A$ is a morphism $f^* : A^* \to B^*$. Moreover, f^* preserves set difference.*

3. The results of Ehrenfeucht and Rozenberg

Given a partial 2-structure $\mathcal{X}$, $\mathsf{Reg}^*_{\mathcal{X}}$ will be of particular interest.

Proposition 3 (*Ehrenfeucht and Rozenberg [2]*). *Let $\mathcal{X} = \langle X, E, \sim\rangle$ be a partial 2-structure. Define a map $\lfloor\ \rfloor_{\mathcal{X}} : X \to 2^{\mathsf{Reg}_{\mathcal{X}}}$ by $\lfloor x\rfloor_{\mathcal{X}} = \{r \in \mathsf{Reg}_{\mathcal{X}} \mid r \ni x\}$. Then it is a morphism of partial 2-structures, $\lfloor\ \rfloor_{\mathcal{X}} : \mathcal{X} \to \mathsf{Reg}^*_{\mathcal{X}}$.*

Definition 8. The regional version of a partial 2-structure $\mathcal{X}$, denoted $\mathsf{RV}\mathcal{X}$, is its image in $\mathsf{Reg}^*_{\mathcal{X}}$ under the morphism $\lfloor\ \rfloor_{\mathcal{X}}$ defined in Proposition 3. The co-restriction of $\lfloor\ \rfloor_{\mathcal{X}}$ on the regional version of $\mathcal{X}$ is denoted by $\mathsf{rv}_{\mathcal{X}} : \mathcal{X} \to \mathsf{RV}\mathcal{X}$.

Definition 9. Given a partial 2-structure $\langle X, E, \sim\rangle$ the following properties, which may or may not hold, are called the *separation axioms*:

- *state-separation*: for any two different states $x, x' \in X$, there exists a region $r : \mathcal{X} \to \mathbf{2}$ such that $rx \neq rx'$,
- *event-separation*: for any two inequivalent edges $\langle x, x'\rangle, \langle y, y'\rangle \in E$, there exists a region r such that exactly one of these edges leaves the region.

Proposition 4 (*Ehrenfeucht and Rozenberg [2, Theorems 5.1, 6.2, and Lemma 6.1]*). *The morphism* $\mathsf{rv}_{\mathcal{X}} : \mathcal{X} \to \mathsf{RV}\mathcal{X}$ *is strongly surjective*; *it is an isomorphism if and only if both separation axioms hold in* $\mathcal{X}$. *The separation axioms are valid in every partial set* 2-*structure* (*hence* $\mathsf{RV}\mathcal{X}$ *and* $\mathsf{RVRV}\mathcal{X}$ *are always isomorphic*); *their validity is preserved under any isomorphism of partial* 2-*structures*.

For instance, the partial set 2-structure $\mathsf{RV}\mathcal{X}$ on the right-hand side of Fig. 1 is the regional version of the partial 2-structure $\mathcal{X}$ on the left-hand side of that figure. It may be checked that both separation axioms hold in $\mathcal{X}$.

The following theorem, due to Ehrenfeucht and Rozenberg, is stated below in a slightly stronger form than in [2]:

Theorem 1 (*Cf. Ehrenfeucht and Rozenberg [2, Theorem 6.1]*). *Given a partial* 2-*structure* $\mathcal{X}$, *a set* B *and a substructure* $\mathcal{Z}$ of B^*, *any morphism of partial* 2-*structures* $\phi : \mathcal{X} \to \mathcal{Z}$ *factors uniquely as* $\phi = \psi \circ \mathsf{rv}_{\mathcal{X}}$, *where* $\mathsf{rv}_{\mathcal{X}} : \mathcal{X} \to \mathsf{RV}\mathcal{X}$ *and* $\psi : \mathsf{RV}\mathcal{X} \to \mathcal{Z}$. *Moreover*, ψ *preserves set difference*.

Proof. Define $\phi_\dagger : B \to 2^X$ as $\phi_\dagger b = \{x \mid \phi x \ni b\}$. Its image lies within $\mathsf{Reg}_{\mathcal{X}}$. Indeed, let $b \in B$ and $x, y, z, w \in X$ be such that $\langle x, y\rangle \sim \langle z, w\rangle$. Suppose that $\langle x, y\rangle$ leaves $\phi_\dagger b$, i.e. $x \in \phi_\dagger b$ and $y \notin \phi_\dagger b$. Then $b \in \phi x \backslash \phi y$. As ϕ is a morphism, $\phi x \vartriangle \phi y = \phi z \vartriangle \phi w$, hence in particular $\phi x \backslash \phi y = \phi z \backslash \phi w$. Therefore, $b \in \phi z \backslash \phi w$, entailing that $z \in \phi_\dagger b$ and $w \notin \phi_\dagger b$, i.e. $\langle z, w\rangle$ leaves $\phi_\dagger b$. One can show similarly that if $\langle x, y\rangle$ enters $\phi_\dagger b$, then $\langle z, w\rangle$ enters $\phi_\dagger b$. By a symmetric reasoning, the converse implications also hold, hence the edges $\langle x, y\rangle$ and $\langle z, w\rangle$ either cross or do not cross the border of $\phi_\dagger b$ in a similar way. As this applies to arbitrary pairs of equivalent edges, $\phi_\dagger b \in \mathsf{Reg}_{\mathcal{X}}$.

Consider $\phi^*_\dagger : \mathsf{Reg}^*_{\mathcal{X}} \to B^*$. Then $\phi^*_\dagger \circ \lfloor\ \rfloor_{\mathcal{X}} = \phi$, for $\phi^*_\dagger(\lfloor x\rfloor_{\mathcal{X}}) = \{b \mid \phi_\dagger b \in \lfloor x\rfloor_{\mathcal{X}}\} = \{b \mid \phi_\dagger b \ni x\} = \{b \mid b \in \phi x\} = \phi x$. Hence its restriction $\psi = \phi^*_\dagger|_{\mathsf{RV}\mathcal{X}}$ sends $\mathsf{RV}\mathcal{X}$ to $\mathcal{Z}$ and fulfils $\psi \circ \mathsf{rv}_{\mathcal{X}} = \phi$ (see Fig. 4). It is uniquely determined because $\mathsf{rv}_{\mathcal{X}}$ is surjective. It preserves set difference as every pre-image function does, hence it is a morphism. $\square$

Corollary 1. *If* $\phi : \mathcal{X} \to \mathcal{Z}$ *is injective* (*resp. strongly injective*), *so are* $\mathsf{rv}_{\mathcal{X}}$ *and* ψ *as defined in Theorem* 1. *If* ϕ *is surjective* (*resp. strongly surjective*), *so is* ψ.

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\phi"] \arrow[dr, "\mathsf{rv}_{\mathcal{X}}"'] & \mathcal{Z} & \subseteq & B^* & & B \arrow[d, "\phi_\dagger"] \\
& \mathsf{RV}\mathcal{X} \arrow[u, "\psi"] & \subseteq & \mathsf{Reg}^*_{\mathcal{X}} \arrow[u, "\phi^*_\dagger"] & & 2^X
\end{tikzcd}$$

Fig. 4.

Note that $\mathsf{rv}_{\mathcal{X}}$ above does not depend on ϕ. So the first statement effectively says that whenever there exists a (strongly) injective ϕ, then $\mathsf{rv}_{\mathcal{X}}$ is (strongly) injective.

Corollary 2. RV *is the object part of a functor that sends any morphism of partial* 2-*structures* $\phi : \mathcal{X} \to \mathcal{Y}$ *to the unique morphism* $\mathsf{RV}\phi : \mathsf{RV}\mathcal{X} \to \mathsf{RV}\mathcal{Y}$ *such that* $\mathsf{rv}_{\mathcal{Y}} \circ \phi = \mathsf{RV}\phi \circ \mathsf{rv}_{\mathcal{X}}$ (*given by Theorem* 1). *Moreover*, RV *is a reflexion, i.e. a left adjoint to an inclusion functor between the category of partial* 2-*structures and the category of partial set* 2-*structures and set difference preserving morphisms. The natural transformation* $\mathsf{rv} : \mathrm{id} \to \mathsf{RV}$ *is the unit of this reflexion*.

Theorem 1 may be summarized by saying that $\mathsf{rv}_{\mathcal{X}} : \mathcal{X} \to \mathsf{RV}\mathcal{X}$ is *initial* among the partial set 2-structure representations of $\mathcal{X}$. An inspection of the proof of Theorem 1 reveals the definition of RV on morphisms. Namely, given $\phi : \mathcal{X} \to \mathcal{Y}$, we see that in the first step one defines $(\mathsf{rv}_{\mathcal{Y}} \circ \phi)_\dagger : \mathsf{Reg}_{\mathcal{Y}} \to \mathsf{Reg}_{\mathcal{X}}$ by $(\mathsf{rv}_{\mathcal{Y}} \circ \phi)_\dagger r = \{x \mid \mathsf{rv}_{\mathcal{Y}}(\phi x) \ni r\} = \{x \mid \phi x \in r\} = \phi^{\leftarrow} r$ and then one shows that its dual $(\mathsf{rv}_{\mathcal{Y}} \circ \phi)^*_\dagger$ restricts, resp. co-restricts, to the regional version of $\mathcal{X}$, resp. $\mathcal{Y}$.

To complete our brief summary of results of Ehrenfeucht and Rozenberg, let us recall the following definition and theorem, adapted from [3]:

Definition 10. A partial set 2-structure $\langle X, E, \sim\rangle$ is *forward-closed* if, for all $z \in X$ and for all $\langle x, y\rangle \in E$ such that $x \backslash y \subseteq z$ and $(y \backslash x) \cap z = \emptyset$, there exists some $w \in X$ such that $w = z \backslash (x \backslash y) \cup (y \backslash x)$ and $\langle z, w\rangle \in E$.

Theorem 2 (*Cf. Ehrenfeucht and Rozenberg [3, Theorem 4.4]*). *A partial* 2-*structure* $\mathcal{X} = \langle X, E, \sim\rangle$ *with a distinguished state* $x_0 \in X$, *seen as a labelled transition system with the initial state* x_0, *is isomorphic to the sequential case graph of a* (*reduced*) *elementary net system iff*

- *every* $x \in X$ *may be reached from* x_0 *following sequences of edges in* E,
- $\mathsf{rv}_{\mathcal{X}}$ *is an isomorphism*,
- $\mathsf{RV}\mathcal{X}$ *is forward-closed*.

We refer the reader to [3] for the definition of the (reduced) elementary net systems and their sequential case graphs. The conditions of Theorem 2 are for instance satisfied in the partial 2-structure $\mathcal{X}$ from Fig. 1.

4. The construction of the closure

Given a partial 2-structure $\mathcal{X}$, the problem we address specifically in this paper is to represent this abstract transition system as the sequential case graph of a (reduced) elementary net system, subject to the additional requirement that this graph should be *complete*. Thus, $\mathcal{X}$ may be represented equally well by the reachability set of the net system, explaining the title of the paper.

Technically speaking, the problem amounts to represent $\mathcal{X}$ as a full and forward-closed substructure of B^* for some set B. As recalled in Section 2, Ehrenfeucht and Rozenberg supplied every partial 2-structure $\mathcal{X}$ with an initial representation $\mathsf{RV}\mathcal{X}$ in the family of all substructures of B^*, for arbitrary (finite) sets B. In this section, we will supply every partial 2-structure $\mathcal{X}$ with an initial representation $\overline{\mathcal{X}}$ in the family of all forward-closed and full substructures of B^*, for arbitrary (finite) sets B.

Theorem 3. *Given a partial* 2*-structure* $\mathcal{X}$, *there exists a forward-closed full substructure* $\overline{\mathcal{X}}$ *of* $\mathsf{Reg}^*_{\mathcal{X}}$ *and a morphism* $\overline{\mathsf{rv}}_{\mathcal{X}} : \mathcal{X} \to \overline{\mathcal{X}}$ *such that, for any set B and for any forward closed full substructure* $\mathcal{Z}$ *of* B^*, *every morphism* $\phi : \mathcal{X} \to \mathcal{Z}$ *factors uniquely as* $\phi = \psi \circ \overline{\mathsf{rv}}_{\mathcal{X}}$, *where* $\psi : \overline{\mathcal{X}} \to \mathcal{Z}$. *Moreover,* ψ *preserves set difference.*

The proof of this theorem will follow directly from the construction of $\overline{\mathcal{X}}$ by an inductive completion of the already defined $\mathsf{RV}\mathcal{X}$ (which may be neither forward closed nor full, as later examples will show), where at each step, new points resp. new edges are inserted in order to comply with the requirement that $\overline{\mathcal{X}}$ should be forward closed and full. Proposition 1 and the lemma below will help us dealing separately at each step with the insertion of edges and with the insertion of points.

Lemma 1. *Let A be a finite set and* $\mathcal{Y} = \langle Y, E, \triangleq\rangle$ *a full substructure of* A^*. *Let* $\mathcal{Y}' = \langle Y', E', \triangleq\rangle$ *be another substructure of* A^*, *extending* $\mathcal{Y}$ (*i.e.* $\mathcal{Y}$ *is a substructure of* $\mathcal{Y}'$) *such that* $(\forall w \in Y' \setminus Y)\ (\exists x, y, z \in Y)\ (\langle z, w\rangle \in E' \wedge \langle z, w\rangle \triangleq \langle x, y\rangle)$. *Let* $\mathcal{Z} = \langle Z, E'', \triangleq\rangle$ *be a forward closed substructure of* B^*. *Then any set difference preserving morphism* $\sigma{:}\mathcal{Y} \to \mathcal{Z}$ *extends uniquely to a morphism* $\sigma' : \mathcal{Y}' \to \mathcal{Z}$. *Moreover,* σ' *preserves set difference.*

Proof. Consider $w \in Y'\backslash Y$ and $x, y, z \in Y$ such that $\langle z, w\rangle \in E'$ and $\langle z, w\rangle \triangleq \langle x, y\rangle$. Thus $x\backslash y \subseteq z$ and $(y\backslash x) \cap z = \emptyset$. As σ preserves set difference it follows that $\sigma x\backslash\sigma y \subseteq \sigma z$ and $(\sigma y\backslash\sigma x)\cap\sigma z = \emptyset$. Define $w' = \sigma z\backslash(\sigma x\backslash\sigma y)\cup(\sigma y\backslash\sigma x)$. If $\sigma x = \sigma y$, then $w' = \sigma z = \sigma w$. In the converse case, $\langle\sigma x, \sigma y\rangle$ is an edge of $\mathcal{Z}$. Since $\mathcal{Z}$ is forward closed, w' is a point of Z, and $\left\langle\sigma z, w'\right\rangle$ is an edge of $\mathcal{Z}$ and is equivalent to $\langle\sigma x, \sigma y\rangle$. Now, any morphism $\sigma' : \mathcal{Y}' \to \mathcal{Z}$ extending $\sigma : \mathcal{Y} \to \mathcal{Z}$ must satisfy $\sigma z \vartriangle \sigma' w = \sigma x \vartriangle \sigma y$, hence $\sigma' w = w'$ in both cases. As σ preserves set difference and $w = z\backslash(x\backslash y) \cup (y\backslash x)$ is in the family of subsets of A generated from Y (using the operations of set difference and disjoint union of sets), it follows that σ' preserves set difference, hence by Proposition 1, σ' is a morphism. □

Proof of the Theorem. Define $\mathcal{X}_0$ as $\mathsf{RV}\mathcal{X}$ in Theorem 1. From this theorem, every morphism $\phi : \mathcal{X} \to \mathcal{Z}$ factors uniquely as $\phi = \psi_0 \circ \mathsf{rv}_{\mathcal{X}}$ and, moreover, the considered ψ_0 preserves set difference. We proceed with an inductive construction (see Fig. 5).

By induction on $n \geqslant 0$, assume $\mathcal{X}_{2n}$ is another substructure of $\mathsf{Reg}^*_{\mathcal{X}}$ extending $\mathcal{X}_0$, such that every morphism $\phi : \mathcal{X} \to \mathcal{Z}$ factors uniquely as $\psi_{2n} \circ (\imath_{2n} \circ \mathsf{rv}_{\mathcal{X}})$, where $\imath_{2n} : \mathcal{X}_0 \to \mathcal{X}_{2n}$ embeds $\mathcal{X}_0$ identically into $\mathcal{X}_{2n}$, and the considered $\psi_{2n} : \mathcal{X}_{2n} \to \mathcal{Z}$ preserves set difference. Let $\mathcal{X}_{2n+1}$ be the full substructure of $\mathsf{Reg}^*_{\mathcal{X}}$ with the same set of

$$\begin{tikzcd}
\mathcal{X} \arrow[rrrr, "\phi"] \arrow[dr, "\mathsf{rv}_{\mathcal{X}}"'] \arrow[drrr, bend right, "\overline{\mathsf{rv}}_{\mathcal{X}}"'] & & & & \mathcal{Z} \subseteq B^* \\
& \mathsf{RV}\mathcal{X} \subseteq \ldots \arrow[urrr, "\psi_0"] & \ldots \subseteq \mathcal{X}_n \ldots \arrow[urr, "\psi_n"] & \subseteq \overline{\mathcal{X}} \arrow[ur, "\psi"] & \subseteq \mathsf{Reg}^*_{\mathcal{X}} \arrow[u, "\phi^*_{\dagger}"']
\end{tikzcd}$$

Fig. 5.

points as $\mathcal{X}_{2n}$. Let ψ_{2n+1} be the map ψ_{2n} (thus it preserves set difference). From Proposition 1, the map ψ_{2n+1} defines a morphism $\psi_{2n+1} : \mathcal{X}_{2n+1} \to \mathcal{Z}$. Obviously, ϕ factors as $\psi_{2n+1} \circ (\imath_{2n+1} \circ \mathsf{rv}_{\mathcal{X}})$, where $\imath_{2n+1}$ embeds $\mathcal{X}_0$ identically into $\mathcal{X}_{2n+1}$, and no other factorization is possible.

Now let $\mathcal{X}_{2n+2}$ be the extension of $\mathcal{X}_{2n+1}$ defined as follows: for every $x, y, z \in X_{2n+1}$ such that $x \backslash y \subseteq z$ and $(y \backslash x) \cap z = \emptyset$, add (if not already present) a new point $w = z \backslash (x \backslash y) \cup (y \backslash x)$ and a new edge $\langle z, w \rangle$. By Lemma 1, ψ_{2n+1} extends to a unique morphism $\psi_{2n+2} : \mathcal{X}_{2n+2} \to \mathcal{Z}$ and preserves set difference. Obviously, ϕ factors as $\psi_{2n+2} \circ (\imath_{2n+2} \circ \mathsf{rv}_{\mathcal{X}})$, where $\imath_{2n+2}$ embeds $\mathcal{X}_0$ identically into $\mathcal{X}_{2n+2}$. As ψ_{2n+2} is the unique extension of ψ_{2n+1}, no other factorization is possible.

Because $\mathsf{Reg}_{\mathcal{X}}$ is a finite set, the inductive construction must reach a step at which $\mathcal{X}_{2n+1} = \mathcal{X}_{2n}$. Then $\overline{\mathcal{X}} = \mathcal{X}_{2n}$, $\overline{\mathsf{rv}}_{\mathcal{X}} = \imath_{2n} \circ \mathsf{rv}_{\mathcal{X}}$ and $\psi = \psi_{2n}$ are like stated in the theorem. □

Corollary 3. *The closure operator is the object part of a functor that sends any morphism of partial* 2*-structures* $\phi : \mathcal{X} \to \mathcal{Y}$ *to the unique morphism* $\overline{\phi} : \overline{\mathcal{X}} \to \overline{\mathcal{Y}}$ *such that* $\overline{\mathsf{rv}}_{\mathcal{Y}} \circ \phi = \overline{\phi} \circ \overline{\mathsf{rv}}_{\mathcal{X}}$ (*given by Theorem* 3). *Moreover, it is a reflexion between the category of partial* 2*-structures and the category of full and forward-closed substructures of set* 2*-structures and set difference preserving morphisms. The natural transformation* $\overline{\mathsf{rv}}$ *is the unit of this reflexion.*

Theorem 3 may be summarized by saying that $\overline{\mathsf{rv}}_{\mathcal{X}} : \mathcal{X} \to \overline{\mathcal{X}}$ is the initial representation of $\mathcal{X}$ by a forward-closed and full substructure of a set 2-structure. Moreover, $\overline{\mathcal{X}}$ is the sequential case graph of an elementary net system (with an arbitrary initial case), since a similar property holds for every forward-closed and full substructure of a set 2-structure. For the sake of an illustration, let $\mathcal{X}$ be the partial 2-structure on the left-hand side of Fig. 1. Then $\mathcal{X}_0$ is the partial set 2-structure $\mathsf{RV}\mathcal{X}$ on the right-hand side of that figure. The partial set 2-structure $\mathcal{X}_0$ is forward-closed, but it is not full. $\mathcal{X}_1$ is the complete graph with the seven nodes of $\mathcal{X}_0$. As was told in the introduction, $\mathcal{X}_1$ is not forward-closed. $\mathcal{X}_2$ has one new node $\{r_1, r_2', r_3\}$, yielding a set of eight nodes in bijection with the product space $\{r_1, r_1'\} \times \{r_2, r_2'\} \times \{r_3, r_3'\}$. $\mathcal{X}_3$ is the complete graph on these eight nodes, and it is forward closed. Thus, $\mathcal{X}_3 = \mathcal{X}_4 = \mathcal{X}_5$ and therefore $\overline{\mathcal{X}} = \mathcal{X}_3$.

In order to conclude the section, let us add a third separation axiom.

Definition 11. Given a partial 2-structure $\mathcal{X} = \langle X, E, \sim \rangle$, the *state-event separation* axiom is valid in $\mathcal{X}$ if, for every edge $\langle x, y \rangle$ and for every state $z \in X$ such that

$\langle x, y \rangle \sim \langle z, w \rangle$ for no $w \in X$, there exists some region $r \in \mathsf{Reg}_{\mathcal{X}}$ such that $rx - ry = 1$ and $rz = 0$.

Let us note that state–event separation of a partial 2-structure $\mathcal{X}$ implies that its regional version $\mathsf{RV}\mathcal{X}$ is forward-closed. If $\mathsf{rv}_{\mathcal{X}}$ is an isomorphism, then the converse implication holds, too.

Theorem 4. *A partial* 2*-structure* $\mathcal{X}$ *is isomorphic to a forward-closed and full substructure of a set* 2*-structure iff it is isomorphic to* $\overline{\mathcal{X}}$ *iff* $\overline{\mathsf{rv}}_{\mathcal{X}}$ *is an isomorphism iff the underlying graph of* $\mathcal{X}$ *is complete and all three separation axioms are valid in* $\mathcal{X}$.

Proof. By Theorem 3, $\mathcal{X}$ is isomorphic to a forward-closed and full substructure of a set 2-structure *iff* $\mathcal{X}$ is isomorphic to $\overline{\mathcal{X}}$ *iff* $\mathcal{X}$ is isomorphic to $\mathsf{RV}\mathcal{X}$ and $\mathsf{RV}\mathcal{X}$ is a forward-closed and full substructure of $\mathsf{Reg}^*_{\mathcal{X}}$ *iff* the underlying graph of $\mathcal{X}$ is complete, $\mathcal{X}$ is isomorphic to $\mathsf{RV}\mathcal{X}$ and $\mathsf{RV}\mathcal{X}$ is forward-closed *iff* the underlying graph of $\mathcal{X}$ is complete, the axioms of state-separation and event-separation are valid in $\mathcal{X}$ (Proposition 4), and the state-event separation axiom is valid in $\mathcal{X}$. $\square$

Corollary 4. $\overline{\mathcal{X}}$ *is isomorphic to* $\overline{\overline{\mathcal{X}}}$.

Proof. All three separation axioms are valid in $\overline{\mathcal{X}}$. $\square$

For a comparison, let us recall the following theorem, adapted from [4]:

Theorem 5 (*Cf. Nielsen [4, Corollary 5.3]*). *A partial* 2*-structure* $\mathcal{X}$ *with initial state* x_0 *is isomorphic to the sequential case graph of an elementary net system iff all states are reachable from* x_0 *and all three separation axioms are valid in* $\mathcal{X}$.

Thus, whenever $\mathcal{X}$ is isomorphic to the sequential case graph of an elementary net system, $\mathcal{X}$ is isomorphic to $\mathsf{RV}\mathcal{X}$ and $\mathsf{RV}\mathcal{X}$ embeds identically into $\overline{\mathcal{X}}$. The conditions of Theorem 5 are satisfied in the partial 2-structure $\mathcal{X}$ from Fig. 1, but the conditions of Theorem 4 are not satisfied, for the underlying graph of $\mathcal{X}$ is not complete. One can see that the regional version of $\mathcal{X}$ ($\mathsf{RV}\mathcal{X}$ in Fig. 1) embeds identically in $\overline{\mathcal{X}}$ (the complete graph with the set of nodes $\{r_1, r_1'\} \times \{r_2, r_2'\} \times \{r_3, r_3'\}$).

5. Properties of the closure

At this stage, the main work yet to be done is to explain the isomorphism $\overline{\mathsf{rv}}_{\overline{\mathcal{X}}}$ between $\overline{\mathcal{X}}$ and $\overline{\overline{\mathcal{X}}}$. For this purpose, we study in this section the connexion between the regions of $\mathcal{X}$ and the regions of $\overline{\mathcal{X}}$. In view of Definition 5 and Theorem 3, the two sets of regions are in bijection. We aim at showing an additional property of this bijection.

Definition 12. Let $\mathcal{X}$ be a partial 2-structure and let $\mathcal{Y}$ be a substructure of $\mathsf{Reg}^*_{\mathcal{X}}$. A map $\phi : \mathsf{Reg}_{\mathcal{Y}} \to \mathsf{Reg}_{\mathcal{X}}$ is a *regional correspondence* between $\mathcal{X}$ and $\mathcal{Y}$ if it is a bijection and $(\forall y \in \mathcal{Y})(\forall R \in \mathsf{Reg}_{\mathcal{Y}})(y \in R \text{ iff } y \ni \phi R)$.

Note that the inverse $\sigma : \mathsf{Reg}_{\mathcal{X}} \to \mathsf{Reg}_{\mathcal{Y}}$ of a regional correspondence ϕ fulfills necessarily $\sigma r = \{y \in \mathcal{Y} \mid y \ni r\}$.

Theorem 6. *For any partial* 2*-structure* $\mathcal{X}$*, there exists a regional correspondence between* $\mathcal{X}$ *and* $\overline{\mathcal{X}}$.

The proof of this theorem follows the inductive pattern of the proof of Theorem 3. We construct a regional correspondence between $\mathcal{X}$ and $\mathcal{X}_0$, and we extend this correspondence inductively, using two different steps according to parity into regional correspondences between $\mathcal{X}_{2n}$ and $\mathcal{X}_{2n+1}$ and between $\mathcal{X}_{2n+1}$ and $\mathcal{X}_{2n+2}$.

Lemma 2. *There exists a regional correspondence between* $\mathcal{X}$ *and* $\mathcal{X}_0$ $(= \mathsf{RV}\mathcal{X})$.

Proof. Define $\phi R = \mathsf{rv}_{\mathcal{X}}{}^{\leftarrow} R$ and $\sigma r = \mathsf{rv}_{\mathcal{X}}{}^{\rightarrow} r$, where R and r are regions of $\mathsf{RV}\mathcal{X}$ and $\mathcal{X}$, respectively. As R is a region and ϕR is its inverse image by a morphism, ϕR is a region (this fact is obvious from Definition 5). As $\mathsf{rv}_{\mathcal{X}}$ is onto, $\sigma r = \{\mathsf{rv}_{\mathcal{X}} x \mid x \in r\} = \{\mathsf{rv}_{\mathcal{X}} x \mid \mathsf{rv}_{\mathcal{X}} x \ni r\} = \{y \in \mathsf{RV}\mathcal{X} \mid y \ni r\}$. To prove that σr is a region, suppose, e.g., $y \in \sigma r$, $y' \notin \sigma r$ and $y \vartriangle y' = z \vartriangle z'$, where $\langle y, y'\rangle$ and $\langle z, z'\rangle$ are edges in $\mathsf{RV}\mathcal{X}$. Then $r \in y \backslash y' = z \backslash z'$, $z \in \sigma r$ and $z' \notin \sigma r$, hence the edge $\langle z, z'\rangle$ crosses σr in the same way as the equivalent edge $\langle y, y'\rangle$.

Again, due to surjectivity of $\mathsf{rv}_{\mathcal{X}}$, $\sigma(\phi R) = R$, hence ϕ and σ are inverses of each other. Thus, $y \in R$ *iff* $y \in \sigma(\phi R)$ *iff* $y \ni \phi R$. $\square$

Lemma 3. *Let* $\mathcal{Y}$ *be a substructure of* $\mathsf{Reg}^*_{\mathcal{X}}$ *and let* $\phi : \mathsf{Reg}_{\mathcal{Y}} \to \mathsf{Reg}_{\mathcal{X}}$ *be a regional correspondence between* $\mathcal{X}$ *and* $\mathcal{Y}$*. Let* $\mathcal{Y}'$ *be a full substructure of* $\mathsf{Reg}^*_{\mathcal{X}}$ *with the same set of points as* $\mathcal{Y}$*. Then* $\mathcal{Y}$ *and* $\mathcal{Y}'$ *have the same set of regions* $(\mathsf{Reg}_{\mathcal{Y}} = \mathsf{Reg}_{\mathcal{Y}'})$ *and* ϕ *is a regional correspondence between* $\mathcal{X}$ *and* $\mathcal{Y}'$.

Proof. Obviously, every region of $\mathcal{Y}'$ is a region of $\mathcal{Y}$, too. On the other hand, let $R \in \mathsf{Reg}_{\mathcal{Y}}$ and suppose, e.g., $y \in R$, $y' \notin R$ and $y \vartriangle y' = z \vartriangle z'$. Then $\phi R \in y \backslash y' = z \backslash z'$, $z \in R$, $z' \notin R$ and thus, the edge $\langle z, z'\rangle$ crosses R in the same way as the equivalent edge $\langle y, y'\rangle$. Hence R is a region of $\mathcal{Y}'$, too. $\square$

Lemma 4. *Let* $\mathcal{Y} = \langle Y, E, \triangleq\rangle$ *be a full substructure of* $\mathsf{Reg}^*_{\mathcal{X}}$ *and let* $\phi : \mathsf{Reg}_{\mathcal{Y}} \to \mathsf{Reg}_{\mathcal{X}}$ *be a regional correspondence between* $\mathcal{X}$ *and* $\mathcal{Y}$*, with inverse* $\sigma : \mathsf{Reg}_{\mathcal{X}} \to \mathsf{Reg}_{\mathcal{Y}}$ *given by* $\sigma r = \{y \in Y \mid y \ni r\}$*. Let* $\mathcal{Y}' = \langle Y', E', \triangleq\rangle$ *be another substructure of* $\mathsf{Reg}^*_{\mathcal{X}}$ *extending* $\mathcal{Y}$ *such that* $(\forall w \in Y' \setminus Y)\ (\exists x, y, z \in Y)\ (\langle z, w\rangle \in E' \wedge \langle z, w\rangle \triangleq \langle x, y\rangle)$*. Then there exists a regional correspondence* ϕ' *between* $\mathcal{X}$ *and* $\mathcal{Y}'$ *and its inverse* $\sigma' : \mathsf{Reg}_{\mathcal{X}} \to \mathsf{Reg}_{\mathcal{Y}'}$ *is given by* $\sigma' r = \{y \in Y' \mid y \ni r\}$.

Proof. Given a region R of $\mathcal{Y}$, define $\varepsilon R = \{y \in Y' \mid y \ni \phi R\}$. Clearly, εR is a region of $\mathcal{Y}'$ and $R = \varepsilon R \cap Y$. We show that εR is the unique region R' of $\mathcal{Y}'$ which extends R. Actually, for every $w \in Y' \backslash Y$, there exist $x, y, z \in Y$ such that $x \vartriangle y = z \vartriangle w$, hence the truth value of the assertion $w \in R'$ is determined by the truth value of the assertions $x \in R'$, $y \in R'$,

$z \in R'$, which are equivalent to the respective assertions $x \in R$, $y \in R$, $z \in R$. Moreover, if one lets $R = \sigma r$, then one may verify that $w \in \varepsilon R$ *iff* $w \ni r$, for w as above. Thus, if we define $\sigma' r = \{y \in Y' \mid y \ni r\}$, then $\varepsilon R = \sigma' r$ for $R = \sigma r$.

On the other hand, the restriction $R' \cap Y$ of a region R' of $\mathcal{Y}'$ is obviously a region of $\mathcal{Y}$ and, by the uniqueness of the extension, $\varepsilon(R' \cap Y) = R'$.

Define $\phi' R' = \phi(R' \cap Y)$, $R' \in \mathsf{Reg}_{\mathcal{Y}'}$. Thus ϕ' is the composition of two bijections. Let $y' \in Y'$ and $R' \in \mathsf{Reg}_{\mathcal{Y}'}$. Then $y' \in R'$ *iff* $y' \in \varepsilon(R' \cap Y)$ *iff* $y' \ni \phi(R' \cap Y)$ *iff* $y' \ni \phi' R'$. Hence ϕ' is a regional correspondence between $\mathcal{X}$ and $\mathcal{Y}'$.

Finally, if $R' = \varepsilon R$ and $R = \sigma r$, then $R' = \sigma' r$ and $\phi' R' = \phi R = r$, and conversely, $\sigma' \phi' R' = \{y' \in Y' \mid y' \ni r\} = \varepsilon R = R'$, thus ϕ' and σ' are inverses. $\square$

Proof of the Theorem. Immediate from Lemmas 2–4 in view of the inductive construction of $\overline{\mathcal{X}}$ in the proof of Theorem 3. $\square$

Corollary 5. *Let $\mathcal{X}$ be a partial 2-structure and let $\phi : \mathsf{Reg}_{\overline{\mathcal{X}}} \to \mathsf{Reg}_{\mathcal{X}}$ be a regional correspondence between $\mathcal{X}$ and $\overline{\mathcal{X}}$. Then $\phi^* : \mathsf{Reg}^*_{\mathcal{X}} \to \mathsf{Reg}^*_{\overline{\mathcal{X}}}$ restricts and corestricts to $\overline{\mathsf{rv}}_{\overline{\mathcal{X}}} : \overline{\mathcal{X}} \to \overline{\overline{\mathcal{X}}}$. Moreover, $\overline{\overline{\mathcal{X}}} = \mathsf{RV}\overline{\mathcal{X}}$ and $\overline{\mathsf{rv}}_{\overline{\mathcal{X}}} = \mathsf{rv}_{\overline{\mathcal{X}}}$.*

Proof. Given $x \in \overline{\mathcal{X}}$, $\phi^* x = \{R \in \mathsf{Reg}_{\overline{\mathcal{X}}} \mid \phi R \in x\} = \{R \in \mathsf{Reg}_{\overline{\mathcal{X}}} \mid R \ni x\} = \overline{\mathsf{rv}}_{\overline{\mathcal{X}}} x \in \overline{\overline{\mathcal{X}}}$. $\square$

Corollary 6. *The isomorphism $\overline{\mathsf{rv}}_{\overline{\mathcal{X}}} : \overline{\mathcal{X}} \to \overline{\overline{\mathcal{X}}}$ maps every point in $\overline{\mathcal{X}}$ to a corresponding point in $\overline{\overline{\mathcal{X}}}$ by simply replacing each token $r \in \mathsf{Reg}_{\mathcal{X}}$ with a corresponding token $\sigma r \in \mathsf{Reg}_{\overline{\mathcal{X}}}$.*

6. Closure adds states to elementary partial 2-structures

The closure operation defined in Section 3 may add states to partial 2-structures even though they are isomorphic to sequential case graphs of elementary net systems. A first example was given in the introduction and commented on in Section 3. In that example, the insertion of the new state $7 = \{r_1, r_2', r_3\}$ followed from taking into account the logical change $\langle\{r_2\}, \{r_2'\}\rangle$ inverse to the logical change $\langle\{r_2'\}, \{r_2\}\rangle$ representing the class of edges b. With respect to the partial 2-structure $\mathcal{X}$ from Fig. 1, $\langle\{r_2\}, \{r_2'\}\rangle$ represents an equivalence class b^{-1} of edges $\langle y, x\rangle$ inverse to the respective edges $\langle x, y\rangle \in b$.

We present in this section another example where the insertion of a new state follows from taking into account the sequential composition of two logical changes. Suppose $\mathcal{X} = \langle X, E, \sim\rangle$ is a partial 2-structure and $\langle x, y\rangle$ and $\langle y, z\rangle$ are two edges of $\mathcal{X}$. Their equivalence classes, call them $a = [\langle x, y\rangle]_\sim$ and $b = [\langle y, z\rangle]_\sim$, are represented by the logical changes $\mathsf{rv}_{\mathcal{X}} x \vartriangle \mathsf{rv}_{\mathcal{X}} y$ and $\mathsf{rv}_{\mathcal{X}} y \vartriangle \mathsf{rv}_{\mathcal{X}} z$, respectively, in the regional version of $\mathcal{X}$. By composing these logical changes sequentially, one obtains the logical change $\mathsf{rv}_{\mathcal{X}} x \vartriangle \mathsf{rv}_{\mathcal{X}} z$. This logical change represents a (possibly new) equivalence class $a{\cdot}b$ of (possibly new) edges, including all pairs $\langle x, z\rangle$ as above.

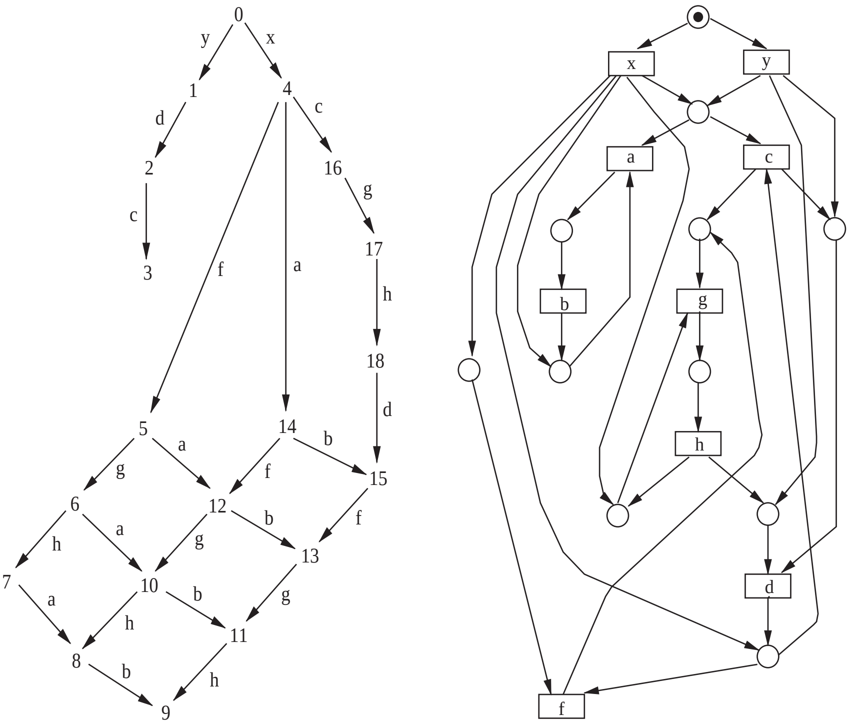

Fig. 6.

Even though the regional version of a partial 2-structure is forward-closed, which must be the case if it fulfils the conditions of Theorem 2 (since then it is isomorphic to the case graph of an elementary net system), the property of forward-closedness may be lost when classes of composite edges are introduced in the partial 2-structure or in its regional version. The closure of the partial 2-structure has then a (strictly) larger set of states.

The simplest example we can bring in support of this claim is depicted on the left-hand side of Fig. 6. A computer tool served us to check the following assertions: The considered graph G, with 0 as the initial state, is isomorphic to the reachable case graph of the elementary net system N depicted on the right-hand side of Fig. 6. In particular, all separation axioms are valid in G. However, if one adds a new class of equivalent edges $a \cdot b$ with $\langle 4, 15 \rangle$, $\langle 5, 13 \rangle$, $\langle 6, 11 \rangle$ and $\langle 7, 9 \rangle$ in this class, then the state-event separation axiom fails for $a \cdot b$ at state 1. Actually, $\overline{G}$ has another edge $\langle 1, 19 \rangle$ in the $(a \cdot b)$-class, where 19 is a new state that differs from all states 1–18, plus many other new states and edges.

The failure of the state-event separation axiom in G with respect to $a \cdot b$ (or equivalently, the failure of the state-event separation axiom in $\mathsf{RV}G$ with respect to the ordered symmetric difference $\mathsf{rv}_G(7) \vartriangle \mathsf{rv}_G(9)$) can be shown as follows: Suppose that $a \cdot b$ exits some region r. Hence, the states 4, 5, 6, 7 are inside r, while 9, 11, 13, 15 are outside of r. Furthermore, the transitions f, g and h do not cross the border of r. Now, the composition $c \cdot g \cdot h \cdot d$ exits r. Hence, either c exits r and d does not cross the border of r, or vice versa. In either case $1 \in r$.

Therefore, if state-event separation holds in G with respect to $a{\cdot}b$, it must be the case that $\langle 1, x\rangle \in a{\cdot}b$, for some state x of G. Exploring further the regions of G, one can see that both sets $\{2, 3, 9, 11, 13, 15\}$ and $\{3, 9, 11, 13, 15, 16, 17, 18\}$ are regions entered by $a{\cdot}b$, hence x must be in their intersection $\{3, 9, 11, 13, 15\}$. The last four states are ruled out, since G is isomorphic to a partial set 2-structure and two edges with a common end point cannot be equivalent in a partial set 2-structure. Finally, $\langle 1, 3\rangle \notin a{\cdot}b$ because $\{3, 5, 12, 13, 16\}$ is a region of G and $a{\cdot}b$ does not cross its border. As a result, $\langle 1, x\rangle \notin a{\cdot}b$ for any node x of G.

The above example shows that the property of forward-closedness may be lost when composite transitions are added, in which case the closure operation actually adds states. Another example presented in the introduction has already shown that the property of forward closedness may be lost by adding inverse transitions. This phenomenon may also be observed in the graph G from Fig. 6. However, one can easily construct from G a larger graph G', which fulfils all conditions of Theorem 2, in which each transition has an inverse and such that forward-closedness is lost when composite transitions are added in G'. Namely, take the net system from Fig. 6, add for each transition in this net an inverse transition and compute the reachable case graph G' of the resulting net. Then G' exhibits the same separation problem as G did with respect to $a{\cdot}b$.

7. A brief comparison with the work of Bernardinello et al.

Let us now explain the relationship between this work and the work presented in [1]. Recall that for any partial 2-structure $\mathcal{X}$ and for any region r in $\mathsf{Reg}_{\mathcal{X}}$, the complement of r is a region of $\mathcal{X}$ (this fact is clear from Definition 5). The representation of $\mathcal{X}$ within $\mathsf{Reg}^*_{\mathcal{X}}$ given in [1] uses the notion of a prime filter. A prime filter in $\mathrm{PF}(\mathsf{Reg}_{\mathcal{X}})$ is any family of regions of $\mathcal{X}$ that contains either r or its complement r', for every $r \in \mathsf{Reg}_{\mathcal{X}}$. Thus, $\mathrm{PF}(\mathsf{Reg}_{\mathcal{X}}) \subseteq \mathsf{Reg}^*_{\mathcal{X}}$ (seen as a set). It is easily shown that also $\mathrm{PF}(\mathsf{Reg}_{\mathcal{X}}) \supseteq \mathsf{RV}\mathcal{X}$ (seen as a set). Indeed, for all $x \in X$ and $r \in \mathsf{Reg}_{\mathcal{X}}$, $r \in \mathsf{rv}_{\mathcal{X}}x$ *iff* $x \in r$ *iff* $x \notin r'$ *iff* $r' \notin \mathsf{rv}_{\mathcal{X}}x$. Further on, one can show that $\mathrm{PF}(\mathsf{Reg}_{\mathcal{X}}) \supseteq \overline{\mathcal{X}}$ (seen as a set). For this purpose, in view of the inductive construction of $\overline{\mathcal{X}}$, it suffices to check that whenever $x, y, z \in \mathrm{PF}(\mathsf{Reg}_{\mathcal{X}})$ and $\langle x, y\rangle \triangleq \langle z, w\rangle$ for some $w \in \mathsf{Reg}^*_{\mathcal{X}}$, this subset w of $\mathsf{Reg}_{\mathcal{X}}$ is a prime filter. If w was not a prime filter, there would exist in $\mathsf{Reg}_{\mathcal{X}}$ two complementary regions r and r' such that both or none of them belongs to w. However, exactly one of the regions r and r' is in z. Suppose, e.g., $r, r' \in w$ and $r' \in z$. Then $r \in w\backslash z = y\backslash x$, and since x and y are prime filters, $r' \in x\backslash y = z\backslash w$, a contradiction. The case $r, r' \notin w$ may be dealt with similarly.

An interesting question about orthomodular posets of regions was left open in [1] (see this reference for precise definitions). Two adjoint functors H and J were defined there, mapping (Condition-Event) transition systems to (Prime Coherent) orthomodular posets and conversely. The functor H maps a transition system to the orthomodular poset formed of its regions. The functor J maps an orthomodular poset to the transition system formed of all prime filters and their ordered symmetric differences. The question is whether either one of the compositions of the two functors is an idempotent functor. This amounts to ask whether there is a regional correspondence between $J \circ H(TS)$ and $(J \circ H) \circ (J \circ H)(TS)$. The regional correspondences which we established between $\mathcal{X}$ and $\overline{\mathcal{X}}$ and between $\overline{\mathcal{X}}$ and

$\overline{\overline{\mathcal{X}}}$ do not help much to solve this problem. Actually, $\overline{\mathcal{X}}$ is a full substructure of $\mathrm{Reg}^*_{\mathcal{X}}$, but it does generally not contain all prime filters, as the following example shows.

Consider the partial 2-structure depicted on the left-hand side of Fig. 3. This partial 2-structure $\mathcal{X}$ (seen as a labelled graph) is isomorphic to the sequential case graph of the elementary net system shown on the right-hand side of Fig. 3. The regional version of this partial 2-structure has six states, namely, the sets of regions $\{r_1, r_2, r_3, r_4\}$, $\{r_1, r_2', r_3', r_4\}$, $\{r_1', r_2, r_3, r_4'\}$, $\{r_1', r_2', r_3', r_4'\}$, $\{r_1, r_2, r_3', r_4'\}$ and $\{r_1', r_2', r_3, r_4\}$. This set of states is closed under all logical changes defined by ordered symmetric differences between states—not only the ones induced by original transitions. Therefore, $\overline{\mathcal{X}}$ is the complete graph with the considered set of nodes. Now, the regions r_1, r_2, r_3', r_4 intersect pairwise, hence if we let X denote the set of states of $\mathcal{X}$, $\{r_1, r_2, r_3', r_4, X\}$ is a prime filter, as observed in [1]. This prime filter is not in $\overline{\mathcal{X}}$. In fact, an even more interesting example of a filter not in $\overline{\mathcal{X}}$ is $\{r_1', r_2, r_3', r_4, X\}$.

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